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CONVERGENCE-SUBTRAHENDS FOR THE TRIGONOMETRICAL FUNCTIONS EXPRESSED IN INFINITE SUMS.

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If $F(z)$ is a function of a complex variable, $z = x + iy$, in general finite and continuous, but infinite on the points, $z = a_1, a_2, a_3, \ldots a_n$, and so infinite that $\lim (z - a_n) F(z) = A_n$ when $z = a_n$, where A_n is not infinite, then

$$F(z) = \sum_1^n \frac{A_n}{z - a_n} + G(z),$$

where $G(z)$ is a function not infinite for any finite value of z. When the number of points on which $F(z)$ is infinite after the manner stated, is infinite, then

$$F(z) = \lim \sum_1^\infty \left(\frac{A_n}{z - a_n} + S(z) \right) + G'(z).^*$$

For any particular function, that which offers the greatest difficulty is the determination of the functions $G(z)$ and $G'(z)$. For this there seems to be no general method, except in so far as the contour-integration method of Cauchy is applicable; and in the application of this, great care is necessary.†

In all cases where an infinite sum is present, $S(z)$, called, conveniently, the Convergence-Subtrahend, must be so determined as to render the infinite sum convergent. Generally a number of functions may be found which will do this, thus making $G'(z)$ also variable. For a new function, after $S(z)$ has been determined, $G'(z)$ is quite arbitrary; for an old function whose properties are well known, $G'(z)$ must be so determined, in connection with the infinite sum, as to bring out these properties.

*See Weierstrass's *Zur Theorie der eindeutigen analytischen Functionen*, Berlin, 1876; the writings of Mittag-Leffler based on this pamphlet; and Schering's *Das Anschliessen einer Function an algebraische Functionen in unendlich vielen Stellen*, Goettingen, 1880.

†Compare Hermite's *Cours 1881–82*, XIIe Leçon, and his note on same in *American Journal of Mathematics* for September, 1883.

The object of this paper is to give a convenient form to $S(z)$ for the functions

$$\frac{1}{\sin z}, \ \frac{1}{\cos z}, \ \tan z, \ \cot z.$$

The only *essentially* singular point which these functions have, is "the point at infinity." Consequently here the infinite sums must be made *absolutely* convergent for all finite values of absolute z, other than those for which the functions become infinite. It will be assumed in what follows, that, for each of these functions, $G'(z)$, in connection with the $S(z)$ which we shall find, is zero. This can be proven, either by contour-integration,* or by showing, on the assumption that $G'(z) = 0$, that the absolutely convergent infinite sums have all the properties of the functions they are made to represent, and agree with them in value, on points selected at random.

$$\frac{1}{\sin z}.$$

$\dfrac{1}{\sin z} = \infty$ when $z = m\pi$, m being 0 or any positive or negative integer, and $\lim \dfrac{z - m\pi}{\sin z} = (-1)^m$ when $z = m\pi$; so that

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_1^\infty \left(\frac{(-1)^m}{z - m\pi} + S_1(z) \right) + \sum_1^\infty \left(\frac{(-1)^m}{z + m\pi} + S_2(z) \right).$$

The sums $\displaystyle\sum_1^\infty \frac{(-1)^m}{z - m\pi}$ and $\displaystyle\sum_1^\infty \frac{(-1)^m}{z + m\pi}$ are not absolutely convergent. The infinite sums in the value of $\dfrac{1}{\sin z}$ are most conveniently rendered absolutely convergent if we determine $S_1(z)$ and $S_2(z)$ so that

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_1^\infty (-1)^m \left(\frac{1}{z - m\pi} + \frac{1}{m\pi} \right) + \sum_1^\infty (-1)^m \left(\frac{1}{z + m\pi} - \frac{1}{m\pi} \right).$$

This is identical in value with the usual expression

$$\frac{1}{\sin z} = \frac{1}{z} + 2z \sum_1^\infty \frac{(-1)^m}{z^2 - m^2\pi^2},$$

but the former shows, almost on its face, the periodicity of $\sin z$; and the periodicity of the trigonometrical functions is their most interesting property.

*See Briot et Bouquet's *Fonctions Doublement Periodiques*, p. 123; also their *Fonctions Elliptiques*, p. 281; also Hermite's *Cours*, p. 79.

$$\frac{1}{\cos z}.$$

$$\frac{1}{\cos z} = \infty \text{ when } z = \pm (2m + 1)\frac{\pi}{2}, \text{ and } \lim \frac{(2m+1)\frac{\pi}{2} - z}{\cos z} = (-1)^m.$$

Hence, corresponding to the formula for $\frac{1}{\sin z}$, we have

$$\frac{1}{\cos z} = \frac{1}{\frac{\pi}{2} - z} + \sum_1^\infty (-1)^m \left\{ \frac{1}{\frac{\pi}{2} - z - m\pi} + \frac{1}{m\pi} \right\}$$
$$+ \sum_1^\infty (-1)^m \left\{ \frac{1}{\frac{\pi}{2} - z + m\pi} - \frac{1}{m\pi} \right\}.$$

This formula shows the periodicity of $\cos z$ and the connection between $\sin z$ and $\cos z$.

Briot and Bouquet, both in their *Fonctions Doublement Periodiques*, p. 125, and in their *Fonctions Elliptiques*, p. 285, obtain, by grouping the terms of an infinite sum, the expression

$$\frac{1}{\cos z} = \pi \sum_0^\infty \frac{(2m+1)(-1)^m}{(2m+1)^2 \frac{\pi^2}{4} - z^2}.$$

This is not an allowable formula, for it is not absolutely convergent for any value of z, as may be readily shown.

Similar formulæ for $\tan z$ and $\cot z$ are

$$\tan z = \frac{1}{\frac{\pi}{2} - z} - \frac{1}{\frac{\pi}{2} + z} + \sum_1^\infty \left\{ \frac{1}{\frac{\pi}{2} - z + m\pi} - \frac{1}{m\pi} \right\} + \sum_1^\infty \left\{ \frac{1}{\frac{\pi}{2} - z - m\pi} + \frac{1}{m\pi} \right\}$$

and

$$\cot z = \frac{1}{z} + \sum_1^\infty \left(\frac{1}{z - m\pi} + \frac{1}{m\pi} \right) + \sum_1^\infty \left(\frac{1}{z + m\pi} - \frac{1}{m\pi} \right).$$